

# INVERSE PROBLEMS, NON-ROUNDNESS AND FLAT PIECES OF THE EFFECTIVE BURNING VELOCITY FROM AN INVISCID QUADRATIC HAMILTON-JACOBI MODEL

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**ABSTRACT.** The main goal of this paper is to understand finer properties of the effective burning velocity from a combustion model introduced by Majda and Souganidis [19]. Motivated by results in [4] and applications in turbulent combustion, we show that when the dimension is two and the flow of the ambient fluid is either weak or very strong, the level set of the effective burning velocity has flat pieces. Due to the lack of an applicable Hopf-type rigidity result, we need to identify the exact location of at least one flat piece. Implications on the effective flame front and other related inverse type problems are also discussed.

## 1. INTRODUCTION

We consider a flame propagation model proposed by Majda, Souganidis [19] described as follows. Suppose that  $V : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a given smooth, mean zero,  $\mathbb{Z}^n$ -periodic and incompressible vector field. Let  $T = T(x, t) : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$  be the solution of the reaction-diffusion-convection equation

$$T_t + V \cdot DT = \kappa \Delta T + \frac{1}{\tau_r} f(T) \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

with given compactly supported initial data  $T(x, 0)$ . Here  $\kappa$  and  $\tau_r$  are positive constants proportional to the flame thickness, which has a small length scale denoted by  $\varepsilon > 0$ . The nonlinear function  $f(T)$  is of KPP type, i.e.,

$$f > 0 \text{ in } (0, 1), \quad f < 0 \text{ in } (-\infty, 0) \cup (1, \infty),$$

$$f'(0) = \inf_{T > 0} \frac{f(T)}{T} > 0.$$

In turbulent combustions, the velocity field usually varies on small scales as well. We write  $V = V\left(\frac{x}{\varepsilon^\gamma}\right)$  and, since the flame thickness is in general much smaller than the turbulence scale, as in [19], we set  $\gamma \in (0, 1)$  and write  $\kappa = d\varepsilon$  and  $\tau_r = \varepsilon$  for some given  $d > 0$ . To simplify notations, throughout this paper, we set  $f'(0) = d = 1$ . The corresponding equation becomes

$$T_t^\varepsilon + V\left(\frac{x}{\varepsilon^\gamma}\right) \cdot DT^\varepsilon = \varepsilon \Delta T^\varepsilon + \frac{1}{\varepsilon} f(T^\varepsilon) \quad \text{in } \mathbb{R}^n \times (0, \infty),$$

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which has a unique solution  $T^\varepsilon$ . It was proven in [19] that  $T^\varepsilon \rightarrow 0$  locally uniformly in  $\{(x, t) : Z < 0\}$ , as  $\varepsilon \rightarrow 0$ , and  $T^\varepsilon \rightarrow 1$  locally uniformly in the interior of  $\{(x, t) : Z = 0\}$ . Here,  $Z \in C(\mathbb{R}^n \times [0, +\infty))$  is the unique viscosity solution of a variational inequality. Moreover, the set  $\Gamma_t = \partial\{x \in \mathbb{R}^n : Z(x, t) < 0\}$  can be viewed as a front moving with the normal velocity:

$$v_{\vec{n}} = \alpha(\vec{n}).$$

Here,  $\alpha$  is the effective burning velocity defined as: For  $p \in \mathbb{R}^n$ ,

$$(1.1) \quad \alpha(p) = \inf_{\lambda > 0} \frac{1 + \overline{H}(\lambda p)}{\lambda}.$$

The convex function  $\overline{H} : \mathbb{R}^n \rightarrow \mathbb{R}$  is called the effective Hamiltonian. For each  $p \in \mathbb{R}^n$ ,  $\overline{H}(p)$  is defined to be the unique constant (ergodic constant) such that the following cell problem

$$(1.2) \quad H(p + Du, x) = |p + Du|^2 + V(x) \cdot (p + Du) = \overline{H}(p) \quad \text{in } \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$$

admits a periodic viscosity solution  $u \in C^{0,1}(\mathbb{T}^n)$ . See [17] for the general statement. As  $\gamma < 1$ , there is no viscous term in (1.2) (see [19, Proposition 1.1]). Under the level-set approach, the effective flame front  $\Gamma_t$  can be described as the zero level set of  $F = F(x, t)$ , which satisfies

$$F_t + \alpha(DF) = 0$$

with  $\Gamma_0 = \{F(x, 0) = 0\}$ . Thus,  $\alpha(p)$  can be viewed as one way to model turbulent flame speed, a significant quantity in turbulent combustion. See [10, 23] for comparisons between  $\alpha(p)$  and the turbulent flame speed modeled by the G-equation (a popular level-set approach model in combustion community).

The original Hamiltonian  $H(p, x) = |p|^2 + V(x) \cdot p$  is similar to the so called Mañé Hamiltonian (or magnetic Lagrangian) in the dynamical system community. Throughout this paper, we assume that  $V$  is smooth,  $\mathbb{Z}^n$ -periodic, incompressible and has mean zero, i.e.,

$$\operatorname{div}(V) = 0 \quad \text{and} \quad \int_{\mathbb{T}^n} V \, dx = 0.$$

It is easy to see that

$$\overline{H}(0) = 0, \quad \overline{H}(p) \geq |p|^2 \quad \text{and} \quad \alpha(p) \geq 2|p|.$$

Moreover,  $\alpha(p)$  is convex and positive homogeneous of degree 1. See Lemma 2.1.

Practically speaking, it is always desirable to get more information of the turbulent flame speed (effective burning velocity). In combustion literature, the turbulent flame speed is often considered to be isotropic and various explicit formulas have been introduced to quantify it. See [1, 2] and the references therein. From the mathematical perspective, it is a very interesting and challenging problem to rigorously identify the shape of the effective Hamiltonian or other effective quantities. In this paper, we are interested in understanding some refined properties of the effective burning velocity  $\alpha(p)$ . In particular,

**Question 1.** If the flow is not at rest, that is, the velocity field  $V$  is not constantly zero, can the convex level set  $\{p \in \mathbb{R}^n : \alpha(p) = 1\}$  be strictly convex? A simpler

inverse type question is whether  $\alpha(p) = c|p|$  for some  $c > 0$  (i.e., isotropic) implies that  $V \equiv 0$ .

**Remark 1.1.** When  $n = 2$ ,  $\alpha(p)$  is actually  $C^1$  away from the origin (Lemma 2.1). If the initial flame front is the circle  $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$ , then, owing to Theorem 2.4, the effective front at  $t > 0$  is  $\Gamma_t = \{x + tD\alpha(x) : x \in S^1\}$  which is a  $C^1$  and strictly convex curve. Obviously, if  $\alpha$  is Euclidean, that is  $\alpha(p) = |p|$ , then  $\Gamma_t$  is round for all  $t > 0$ . If the level curve  $\{\alpha(p) = 1\}$  contains a flat piece, then there exist  $x_0, x_1 \in S^1$  such that

$$D\alpha(x_s) \equiv D\alpha(x_0) \quad \text{for } x_s = (1-s)x_0 + sx_1 \text{ and } s \in [0, 1].$$

In view of the positive homogeneity of  $\alpha$ ,  $D\alpha(p) = D\alpha(x_0) = D\alpha(x_1)$  for all  $p \in S^1$  between  $x_0$  and  $x_1$ , i.e.  $(p - x_0) \cdot (p - x_1) < 0$ . Owing to the representation of  $\Gamma_t$ , the arc between  $x_0$  and  $x_1$  of  $S^1$  is translated in time and is contained in the front  $\Gamma_t$ . This sort of implies that along the direction  $D\alpha(x_0)$ , the linear transport overwhelms the nonlinear reaction term and dominates the spread of flame, i.e., the propagation behaves like  $F_t + D\alpha(x_0) \cdot DF = 0$ .

Before stating the main results, we review some related works that partly motivate the study of the above questions from the mathematical perspective. Consider the metric Hamiltonian  $H(p, x) = \sum_{1 \leq i, j \leq n} a_{ij}(x)p_i p_j$  with smooth, periodic and positive definite coefficient  $(a_{ij})$ . It was proven in a very interesting paper of Bangert [4] that, for  $n = 2$ , if the convex level curve  $\{p \in \mathbb{R}^2 : \overline{H}(p) \leq 1\}$  is strictly convex, then  $(a_{ij})$  must be a constant matrix. The argument consists of two main ingredients. First, through a delicate analysis using two dimensional topology, Bangert showed that if the level set is strictly convex at a point, then the corresponding Mather set of that point is the whole torus  $\mathbb{T}^2$  and it is foliated by minimizing geodesics pointing to a specific direction. Secondly, a well-known theorem of Hopf [16] was then applied which says that a periodic Riemannian metric on  $\mathbb{R}^2$  without conjugate points must be flat. Part of Bangert's results (e.g. foliation of the 2d torus by minimizing orbits) was extended to Tonelli Hamiltonians in [20] for more general surfaces. Combining with the Hopf-type rigidity result in [6] for magnetic Hamiltonian, when  $n = 2$ , it is easy to derive that the level set of the  $\overline{H}$  associated with the Mañé-type Hamiltonian (1.2) must contain flat pieces unless  $V \equiv 0$ . We would like to point out that the non-strict convexity has not been established for general Tonelli Hamiltonian due to the lack of Hopf's rigidity result for Finsler metrics. See [24, 22] for instance.

The main difficulty in our situation is that the effective burning velocity  $\alpha$  is related to the effective Hamiltonian  $\overline{H}$  through a variational formulation; see (1.1). In particular, the level set of  $\alpha$  is not the same as that of  $\overline{H}$  and Hopf-type rigidity results are not applicable. In contrast to the proof in [4], we need to figure out the exact location of at least one flat piece in our proofs, which is of independent interest.

In this paper, we establish some results concerning Question 1 when the flow is either very weak or very strong. The first theorem is for any dimension.

**Theorem 1.1.** Assume that  $V$  is not constantly zero. Then there exists  $\varepsilon_0 > 0$  such that when  $\varepsilon \in (0, \varepsilon_0)$ , the level curve  $S_\varepsilon = \{p \in \mathbb{R}^n : \alpha_\varepsilon(p) = 1\}$  is not round

(or equivalently, the function  $\alpha_\varepsilon$  is not Euclidean). Here,  $\alpha_\varepsilon$  is the effective burning velocity associated with the flow velocity  $\varepsilon V$ .

In two dimensional space, thanks to Lemma 2.1,  $\alpha(p) \in C^1(\mathbb{R}^2 \setminus \{0\})$ . We prove further that the level curve of  $\alpha$  is not strictly convex under small or strong advections. To state the result precisely, we recall that, for a set  $S \subset \mathbb{R}^n$ , a point  $p$  is said to be a *linear point* of  $S$  if there exists a unit vector  $q$  and a positive number  $\mu_0 > 0$  such that the line segment  $\{p + tq : t \in [0, \mu_0]\} \subseteq S$ .

**Theorem 1.2.** *Assume that  $n = 2$  and  $V$  is not constantly zero. Then*

- (1) (weak flow) *there exists  $\varepsilon_0 > 0$  such that when  $\varepsilon \in (0, \varepsilon_0)$ , the level curve  $S_\varepsilon = \{p \in \mathbb{R}^2 : \alpha_\varepsilon(p) = 1\}$  contains flat pieces. Here,  $\alpha_\varepsilon$  is the effective burning velocity associated with the flow velocity  $\varepsilon V$ .*
- (2) (strong flow) *there exists  $A_0 > 0$  such that when  $A \geq A_0$ , the level curve  $S_A = \{p \in \mathbb{R}^2 : \alpha_A(p) = 1\}$  contains flat pieces. Here,  $\alpha_A$  is the effective burning velocity associated with the flow velocity  $AV$ . In particular, if the flow  $\dot{\xi} = V(\xi)$  has a swirl (i.e., a closed orbit that is not a single point), any  $p \in S_A$  which has a rational outward normal vector is a linear point of  $S_A$  when  $A \geq A_0$ .*

We conjecture that the above theorem holds for all amplitude parameters  $A \in (0, \infty)$ . So far, we can only show this for some special flows. Precisely speaking,

**Theorem 1.3.** *Assume either*

- (1) (shear flow)  $V(x) = (v(x_2), 0)$  for  $x = (x_1, x_2) \in \mathbb{R}^2$ , where  $v : \mathbb{R} \rightarrow \mathbb{R}$  is a 1-periodic smooth function with mean zero, or
- (2) (cellular flow)  $V(x) = (-K_{x_2}, K_{x_1})$  with  $K(x_1, x_2) = \sin(2\pi x_1) \sin(2\pi x_2)$  for  $x = (x_1, x_2) \in \mathbb{R}^2$ .

*Then for any fixed  $A \neq 0$ , the level curve  $S_A = \{p \in \mathbb{R}^2 : \alpha_A(p) = 1\}$  contains flat pieces. Here,  $\alpha_A$  is the effective burning velocity associated with the flow velocity  $AV$ .*

We would like to point out that for the cellular flow in part (2) of Theorem 1.3, it was derived by Xin and Yu [23] that

$$\lim_{A \rightarrow +\infty} \frac{\log(A) \alpha_A(p)}{A} = C(|p_1| + |p_2|).$$

for  $p = (p_1, p_2) \in \mathbb{R}^2$  and a fixed constant  $C$ . See Remark 2.1 for front motion associated with the Hamiltonian  $H(p) = |p_1| + |p_2|$ . Moreover, it remains an interesting question to at least extend the above global result to flows which have both shear and cellular structures, e.g., the cat's eye flow. A prototypical example is  $V(x) = (-K_{x_2}, K_{x_1})$  with  $K(x_1, x_2) = \sin(2\pi x_1) \sin(2\pi x_2) + \delta \cos(2\pi x_1) \cos(2\pi x_2)$  for  $\delta \in (0, 1)$ .

**General inverse problems.** In general, the effective burning velocity cannot determine the structure of the ambient fluid. The reason is that the function  $\alpha(p)$  is homogeneous of degree one and only depends on the value of  $\overline{H}$  from (1.2) in a bounded domain. So  $\alpha(p)$  cannot see the velocity field  $V$  in places where it rotates very fast. See Claim 1 in the proof of Theorem 1.2. See also (9.5) in [3] for a related

situation. Nevertheless, we can address the following inverse type problem for the effective Hamiltonian  $\overline{H}(p)$ .

**Question 2.** Assume that  $H_i(p, y) = |p|^2 + V_i(y) \cdot p$ . Assume further that  $\overline{H}_1 = \overline{H}_2$ , where  $\overline{H}_i$  is the corresponding effective Hamiltonian of  $H_i$  for  $i = 1, 2$ . Then what can we conclude about the relations between  $V_1$  and  $V_2$ ? Especially, can we identify some common “computable” properties shared by  $V_1$  and  $V_2$ ?

This kind of questions was posed and studied first in Luo, Tran and Yu [18] for Hamiltonians of separable forms, i.e., when  $H_i(p, y) = H(p) + W_i(y)$  for  $i = 1, 2$ . Here  $H(p)$  is the kinetic energy and  $W_i$  is the potential energy. As discussed in [18], a lot of tools from dynamical systems, e.g. KAM theory, Aubry-Mather theory, are involved in the study and the analysis of the problems. Using the approach of “asymptotic expansion at infinity” introduced in [18], we can show that if the Fourier coefficients of  $V_i$  for  $i = 1, 2$  decay very fast, then

$$\overline{H}_1 = \overline{H}_2 \quad \Rightarrow \quad \int_{\mathbb{T}^n} |V_1|^2 dy = \int_{\mathbb{T}^n} |V_2|^2 dy.$$

Since the proof is similar to that of (3) in Theorem 1.2 of [18], we omit it here.

**Outline of the paper.** For readers’ convenience, we give a quick review of Mather sets and the weak KAM theory in Section 2. Some basic properties of  $\alpha(p)$  (e.g. the  $C^1$  regularity) will be derived as well. In Section 3, we prove Theorems 1.1 via perturbation arguments. Theorems 1.2 and 1.3 will be established in Section 4. The use of two dimensional topology is extremely essential here and we do not know yet if the results of Theorems 1.2 and 1.3 can be extended to higher dimensional spaces.

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## 2. PRELIMINARIES

For the reader’s convenience, we briefly review some basic results concerning the Mather sets and the weak KAM theory. See [9, 11, 13] for more details. Let  $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$  be the  $n$ -dimensional flat torus and  $H(p, x) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$  be a Tonelli Hamiltonian, i.e., it satisfies that

- (H1) (Periodicity)  $x \mapsto H(p, x)$  is  $\mathbb{T}^n$ -periodic;
- (H2) (Uniform convexity) There exists  $c_0 > 0$  such that for all  $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$ , and  $(p, x) \in \mathbb{R}^n \times \mathbb{R}^n$ ,

$$\sum_{i,j=1}^n \eta_i \frac{\partial^2 H}{\partial p_i \partial p_j} \eta_j \geq c_0 |\eta|^2.$$

Let  $L(q, x) = \sup_{p \in \mathbb{R}^n} \{p \cdot q - H(p, x)\}$  be the Lagrangian associated with  $H$ . Let  $\mathcal{W}$  denote the set of all Borel probability measures on  $\mathbb{R}^n \times \mathbb{T}^n$  that are invariant under the corresponding Euler-Lagrangian flow.

For each fixed  $p \in \mathbb{R}^n$ , an element  $\mu$  in  $\mathcal{W}$  is called a Mather measure if

$$\int_{\mathbb{R}^n \times \mathbb{T}^n} (L(q, x) - p \cdot q) d\mu = \min_{\nu \in \mathcal{W}} \int_{\mathbb{R}^n \times \mathbb{T}^n} (L(q, x) - p \cdot q) d\nu,$$

that is, if it minimizes the action associated to  $L(q, x) - p \cdot q$ . Denote by  $\mathcal{W}_p$  the set of all such Mather measures. The value of the minimum action turns out to be  $-\overline{H}(p)$ , where  $\overline{H}(p)$  is the unique real number such that the following Hamilton-Jacobi equation

$$(2.3) \quad H(p + Du, x) = \overline{H}(p) \quad \text{in } \mathbb{T}^n$$

has a periodic viscosity solution  $u \in C^{0,1}(\mathbb{T}^n)$ . Equation (2.3) is usually called the cell problem and  $\overline{H}$  is called the effective Hamiltonian.

The Mather set is defined to be the closure of the union of the support of all Mather measures, i.e.,

$$\widetilde{\mathcal{M}}_p = \overline{\bigcup_{\mu \in \mathcal{W}_p} \text{supp}(\mu)}.$$

The projected Mather set  $\mathcal{M}_p$  is the projection of  $\widetilde{\mathcal{M}}_p$  to  $\mathbb{T}^n$ . The following basic and important properties of the Mather set are used frequently in this paper.

(1) For any viscosity solution  $u$  of equation (2.3), we have that

$$(2.4) \quad \widetilde{\mathcal{M}}_p \subset \{(q, x) \in \mathbb{R}^n \times \mathbb{T}^n : Du(x) \text{ exists and } p + Du(x) = D_q L(q, x)\}.$$

Moreover  $u \in C^{1,1}(\mathcal{M}_p)$ . More precisely, there exists a constant  $C$  depending only on  $H$  and  $p$  such that, for all  $y \in \mathbb{T}^n$  and  $x \in \mathcal{M}_p$ ,

$$\begin{aligned} |u(y) - u(x) - Du(x) \cdot (y - x)| &\leq C|y - x|^2, \\ |Du(y) - Du(x)| &\leq C|y - x|. \end{aligned}$$

(2) For any orbit  $\xi : \mathbb{R} \rightarrow \mathbb{T}^n$  such that  $(\dot{\xi}(t), \xi(t)) \in \widetilde{\mathcal{M}}_p$  for all  $t \in \mathbb{R}$ , we lift  $\xi$  to  $\mathbb{R}^n$  and denote the lifted orbit on  $\mathbb{R}^n$  still by  $\xi$ . Then,  $\xi$  is an absolutely minimizing curve with respect to  $L(q, x) - p \cdot q + \overline{H}(p)$  in  $\mathbb{R}^n$ , i.e., for any  $-\infty < s_2 < s_1 < \infty$ ,  $-\infty < t_2 < t_1 < \infty$  and  $\gamma : [s_2, s_1] \rightarrow \mathbb{R}^n$  absolutely continuous satisfying  $\gamma(s_2) = \xi(t_2)$  and  $\gamma(s_1) = \xi(t_1)$  the following inequality holds,

$$(2.5) \quad \int_{s_1}^{s_2} (L(\dot{\gamma}(s), \gamma(s)) + \overline{H}(p)) ds \geq \int_{t_1}^{t_2} (L(\dot{\xi}(t), \xi(t)) + \overline{H}(p)) dt.$$

Moreover, if  $\xi$  is a periodic orbit, then its rotation vector

$$(2.6) \quad \frac{\xi(T) - \xi(0)}{T} \in \partial \overline{H}(p).$$

Here  $T$  is the period of  $\xi$  and  $\partial \overline{H}(p)$  is the subdifferential of  $\overline{H}$  at  $p$ , i.e.,  $q \in \partial \overline{H}(p)$  if  $\overline{H}(p') \geq \overline{H}(p) + q \cdot (p' - p)$  for all  $p' \in \mathbb{R}^n$ .

A central problem in weak KAM theory is to understand the relation between analytic properties of the effective Hamiltonian  $\overline{H}$  and the underlying Hamiltonian system (e.g. structures of Mather sets). For instance, Bangert [3] gave a detailed characterization of Mather and Aubry sets on the 2-torus  $\mathbb{T}^2$  for metric or mechanical Hamiltonians (i.e.,  $H(p, x) = \sum_{1 \leq i, j \leq n} a_{ij} p_i p_j + W(x)$ ).

Let us mention some known results in this direction which are more relevant to this paper. As an immediate corollary of [7, Proposition 3], we have the following result concerning the level curves of  $\overline{H}$  in two dimensional space.

**Theorem 2.1.** *Assume that  $n = 2$ . If  $\overline{H}(p) = c > \min \overline{H}$ , then the set  $\partial \overline{H}(p)$  is a closed radial interval, i.e., there exist a unit vector  $q \in \mathbb{R}^2$  and  $0 < s_1 \leq s_2$  such that  $\partial \overline{H}(p) = [s_1 q, s_2 q]$ . In particular, this implies that the level set  $\{p \in \mathbb{R}^2 : \overline{H}(p) = c\}$  is a closed  $C^1$  convex curve and  $q$  is the unit outward normal vector at  $p$ .*

The following theorem was first proven in [11, Theorem 8.1]. It says that the effective Hamiltonian is strictly convex along any direction that is not tangent to the level set.

**Theorem 2.2.** *Assume that  $p_1, p_2 \in \mathbb{R}^n$ . Suppose that  $\overline{H}(p_2) > \min \overline{H}$  and  $\overline{H}$  is linear along the line segment connecting  $p_1$  and  $p_2$ . Then*

$$\overline{H}(tp_1 + (1-t)p_2) \equiv \overline{H}(p_2) \quad \text{for all } t \in [0, 1].$$

In dynamical system literature, the effective Hamiltonian  $\overline{H}$  and its Lagrangian  $\overline{L}$  are often called  $\alpha$  and  $\beta$  functions respectively. Since  $Q \in \partial \overline{H}(P) \Leftrightarrow P \in \partial \overline{L}(Q)$ , that  $\overline{L}$  is not differentiable at  $Q$  implies that  $\overline{H}$  is linear along any two vectors in  $\partial \overline{L}(Q)$ . Accordingly, as an immediate outcome of [20, Corollary 1], we have that

**Theorem 2.3.** *Let  $n = 2$ ,  $c > \min \overline{H}$  and  $p \in \Gamma_c = \{\overline{H} = c\}$ . If the unit normal vector of  $\Gamma_c$  at  $p$  is a rational vector and  $p$  is not a linear point of  $\Gamma_c$ , then  $\mathcal{M}_p$  consists of periodic orbits which foliate  $\mathbb{T}^2$ .*

For metric or mechanical Hamiltonians (i.e.,  $H(p, x) = \sum_{1 \leq i, j \leq n} a_{ij} p_i p_j + W(x)$ ), the above result was first established in [4].

Finally, we prove some simple properties of the effective burning velocity  $\alpha(p)$ .

**Lemma 2.1.** *Fix  $p \in \mathbb{R}^n \setminus \{0\}$ . The followings hold*

- (1)  $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$  is convex. Also, there exists a unique  $\lambda_p > 0$  such that

$$\alpha(p) = \frac{1 + \overline{H}(\lambda_p p)}{\lambda_p}.$$

Moreover, there exists  $q \in \partial \overline{H}(\lambda_p p)$  such that

$$q \cdot \lambda_p p = \overline{H}(\lambda_p p) + 1.$$

- (2) Assume that  $n = 2$ . Then  $\alpha(p) \in C^1(\mathbb{R}^n \setminus \{0\})$ .  
 (3) Assume that  $n = 2$ . Then  $p$  is a linear point of the level curve  $\{\alpha = 1\}$  if and only if  $\lambda_p p$  is a linear point of the level curve  $\{\overline{H} = \lambda_p - 1\}$ .

*Proof.* (1) Since  $\overline{H}(p) \geq |p|^2$ , the existence of  $\lambda_p$  is clear. For the convexity of  $\alpha$ , fix  $p_0, p_1 \in \mathbb{R}^n \setminus \{0\}$  and choose  $\lambda_0, \lambda_1 > 0$  such that

$$\alpha(p_0) = \frac{1 + \overline{H}(\lambda_0 p_0)}{\lambda_0} \quad \text{and} \quad \alpha(p_1) = \frac{1 + \overline{H}(\lambda_1 p_1)}{\lambda_1}.$$

For  $\theta \in [0, 1]$ , write  $p_\theta = \theta p_1 + (1 - \theta)p_0$ . If  $p_\theta = 0$ , the convexity is obvious since  $\alpha(0) = 0$  and  $\alpha(p) \geq 2|p|$ . So we assume  $p_\theta \neq 0$ . Choose  $\lambda_\theta > 0$  such that

$\frac{1}{\lambda_\theta} = \frac{\theta}{\lambda_1} + \frac{1-\theta}{\lambda_0}$ . It follows immediately from the definition of  $\alpha$  and the convexity of  $\bar{H}$  that

$$\alpha(p_\theta) \leq \frac{1 + \bar{H}(\lambda_\theta p_\theta)}{\lambda_\theta} \leq \theta \alpha(p_1) + (1 - \theta) \alpha(p_0).$$

The convexity of  $\alpha$  is proved.

Next we prove the uniqueness of  $\lambda_p$ . Assume that for  $\lambda, \bar{\lambda} > 0$ , we have that

$$\alpha(p) = \frac{1 + \bar{H}(\lambda p)}{\lambda} = \frac{1 + \bar{H}(\bar{\lambda} p)}{\bar{\lambda}}.$$

Then  $\partial\alpha(p) \subseteq \partial\bar{H}(\lambda p)$  and  $\partial\alpha(p) \subseteq \partial\bar{H}(\bar{\lambda} p)$ . Therefore,  $\partial\bar{H}(\lambda p) \cap \partial\bar{H}(\bar{\lambda} p) \neq \emptyset$ . So  $\bar{H}$  is linear along the line segment connecting  $\lambda p$  and  $\bar{\lambda} p$ . Then by Theorem 2.2,  $\bar{H}(\lambda p) = \bar{H}(\bar{\lambda} p)$ , which immediately leads to  $\lambda = \bar{\lambda}$ .

Next we prove the second equality in Claim (1). For  $\lambda > 0$ , denote by  $w(\lambda) = \bar{H}(\lambda p) \geq \lambda^2 |p|^2$  and

$$h(\lambda) = \frac{1 + w(\lambda)}{\lambda}.$$

Since  $w(\lambda)$  is convex, there exists a decreasing sequence  $\{\lambda_m\}$  such that  $\lambda_m \downarrow \lambda_p$  and  $w$  is differentiable at  $\lambda_m$  and  $h'(\lambda_m) \geq 0$ . Clearly,

$$w'(\lambda_m) = q_m \cdot p \quad \text{for any } q_m \in \partial\bar{H}(\lambda_m p).$$

Up to a subsequence, we may assume that  $q_m \rightarrow q^+ \in \partial\bar{H}(\lambda_p p)$ . Then in light of the fact that  $h'(\lambda_m) \geq 0$ , we deduce

$$q^+ \cdot \lambda_p p \geq \bar{H}(\lambda_p p) + 1.$$

Similarly, by considering an increasing sequence that converges to  $\lambda_p$ , we can pick  $q^- \in \partial\bar{H}(\lambda_p p)$  such that

$$q^- \cdot \lambda_p p \leq \bar{H}(\lambda_p p) + 1.$$

Since  $\partial H(\lambda_p p)$  is a convex set, we can find  $q \in \partial\bar{H}(\lambda_p p)$  which satisfies

$$q \cdot \lambda_p p = \bar{H}(\lambda_p p) + 1.$$

(2) Apparently,

$$(2.7) \quad \hat{q} \in \partial\alpha(p) \quad \Rightarrow \quad \hat{q} \in \partial\bar{H}(\lambda_p p).$$

Owing to Theorem 2.1,  $\partial\alpha(p)$  is also a closed radial interval. Since  $\alpha(p)$  is homogeneous of degree 1, any  $q \in \partial\alpha(p)$  satisfies  $p \cdot q = \alpha(p)$ . Since  $p \neq 0$  and  $\alpha(p) > 0$ , this interval can only contain a single point; it follows that  $\alpha$  is differentiable at  $p$ .

(3) “ $\Rightarrow$ ”: This part is true in any dimension. Clearly, that  $p$  is a linear point of  $S = \{\alpha = 1\}$  implies that there exists  $p' \in S$  such that  $p \neq p'$  and

$$\partial\alpha(p) \cap \partial\alpha(p') \neq \emptyset.$$

By (2.7),  $\partial\alpha(p) \subseteq \partial\bar{H}(\lambda_p p)$  and  $\partial\alpha(p') \subseteq \partial\bar{H}(\lambda_{p'} p')$ . Hence  $\bar{H}$  is linear along the line segment connecting  $\lambda_p p$  and  $\lambda_{p'} p'$ . Then Theorem 2.2 implies that  $\bar{H}(\lambda_p p) = \bar{H}(\lambda_{p'} p')$  and  $\lambda_p = \lambda_{p'}$ . The necessity then follows.

Now we prove the sufficiency which relies on the 2-dimensional topology. For  $p \in \mathbb{R}^2$ , assume that  $\lambda_p p$  is a linear point of the level curve  $C_p = \{\bar{H} = \lambda_p - 1\}$ , i.e., there exists a distinct vector  $\lambda_p p' \in C_p$  such that the line segment  $l_{p,p'} =$



$\{sp + (1-s)p' : s \in [0, 1]\}$ , which connects  $p$  and  $p'$ , satisfies  $l_{p,p'} \subset \{G = 1\}$ . Here for  $q \in \mathbb{R}^2$ ,

$$G(q) = \frac{1 + \overline{H}(\lambda_p q)}{\lambda_p} \geq \alpha(q).$$

By Theorem 2.1 and  $D\alpha(p) \in \partial G(p) = \partial \overline{H}(\lambda_p p)$ , we have that

$$\partial G(p) = \{sD\alpha(p) : s \in [\theta_1, \theta_2]\}$$

for some  $0 < \theta_1 \leq \theta_2$ . Therefore  $D\alpha(p) \cdot (p' - p) = 0$ , which implies that

$$1 = G(q) \geq \alpha(q) \geq \alpha(p) + D\alpha(p) \cdot (q - p) = \alpha(p) = 1$$

for any  $q \in l_{p,p'}$ . Hence  $l_{p,p'} \subset \{\alpha = 1\}$  and  $p$  is a linear point. □

The following result characterizes the shape of the moving front when the initial front is the unit circle in  $\mathbb{R}^2$ .

**Theorem 2.4.** *Suppose that  $n = 2$  and  $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}$  is convex, coercive and positive homogeneous of degree 1. Let  $u \in C(\mathbb{R}^2 \times [0, +\infty))$  be the unique viscosity solution to*

$$\begin{cases} u_t + \alpha(Du) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\ u(x, 0) = |x| - 1. \end{cases}$$

*Then  $u(x, t) = \max\{-t\alpha(p) + x \cdot p : |p| \leq 1\} - 1$  and its zero level set is*

$$(2.8) \quad \Gamma_t := \{x \in \mathbb{R}^2 : u(x, t) = 0\} = \{p + tq : p \in S^1, q \in \partial\alpha(p)\}.$$

*Also  $\Gamma_t$  is  $C^1$ . Moreover,*

$$(2.9) \quad \alpha \in C^1(\mathbb{R}^2 \setminus \{0\}) \iff \Gamma_t \text{ is strictly convex.}$$

*Proof.* We first prove the representation (2.8). Due to the 1-homogeneity of  $\alpha(p)$ ,  $p \cdot q = \alpha(p)$  for any  $q \in \partial\alpha(p)$ . The formula of  $u(x, t)$  then follows directly from Theorem 3.1 in [5]. Clearly, if  $u(x, t) > -1$ , then

$$u(x, t) = \max\{-t\alpha(p) + x \cdot p : |p| = 1\} - 1.$$

Now fix  $x \in \mathbb{R}^2$  such that  $u(x, t) = 0$ . Choose  $|\bar{p}| = 1$  such that

$$(2.10) \quad u(x, t) = \bar{p} \cdot x - t\alpha(\bar{p}) - 1 = 0.$$

By the Lagrange multiplier method, we get  $x - tq = s\bar{p}$  for some  $q \in \partial\alpha(\bar{p})$  and some  $s \in \mathbb{R}$ . We use (2.10) to deduce further that  $s = 1$ , and hence  $x = \bar{p} + tq$ .

Conversely, if  $x = \bar{p} + tq$  for some  $\bar{p} \in S^1$  and  $q \in \partial\alpha(\bar{p})$ , we want to show that  $u(x, t) = 0$ . In fact, in the representation formula of  $u$ , choosing  $p = \bar{p}$  immediately leads to  $u(x, t) \geq 0$ . On the other hand, for any  $|p| = 1$ ,  $q \in \partial\alpha(\bar{p})$  implies

$$\alpha(p) \geq \alpha(\bar{p}) + q \cdot (p - \bar{p}).$$

Therefore

$$-t\alpha(p) + x \cdot p \leq -t\alpha(\bar{p}) - tq \cdot (p - \bar{p}) + x \cdot p = p \cdot \bar{p} \leq 1.$$

So  $u(x, t) \leq 0$ . Hence we proved that  $u(x, t) = 0$ .

Next we show that  $\Gamma_t$  is  $C^1$ . Fix  $t > 0$ . Owing to the above arguments, given  $x \in \Gamma_t$ , there exists a unique unit vector  $p_x$  such that  $x = p_x + q_x$  for some  $q_x \in \partial\alpha(p_x)$  and

$$u(x, t) = -t\alpha(p_x) + p_x \cdot x - 1.$$

The uniqueness is due to the convexity of  $\alpha$  which implies that  $(p - p') \cdot (q - q') \geq 0$  for  $q \in \partial\alpha(p)$  and  $q' \in \partial\alpha(p')$ . Hence  $x \rightarrow p_x$  is a continuous map from  $\Gamma_t$  to the unit circle. Combining with  $p_x \in \partial_x u(x, t)$ ,  $p_x$  is the outward unit normal vector of  $\Gamma_t$  at  $x$  and  $\Gamma_t$  is  $C^1$ .

Next we prove the duality (2.9). Again fix  $t > 0$ . This direction “ $\Leftarrow$ ” follows immediately from the representation formula (2.8). So let us prove “ $\Rightarrow$ ”. We argue by contradiction. Assume that  $\alpha$  is  $C^1$  away from the origin. If  $\Gamma_t$  is not strictly convex, then there exist  $x, y \in \Gamma_t$  such that  $x \neq y$  and  $p_x = p_y$ . Hence  $q_x \neq q_y$ . However,  $q_x = D\alpha(p_x) = D\alpha(p_y) = q_y$ , which is a contradiction. This proves that (2.9) holds.  $\square$

**Remark 2.1.** As mentioned in Remark 1.1, when  $n = 2$ , a flat piece on the level set  $\{\alpha(p) = 1\}$  leads to a translated arc of the unit circle on  $\Gamma_t$ . Moreover, singular points of  $\alpha$  (i.e., points where  $\partial\alpha(p)$  contains a line segment) generate flat pieces on  $\Gamma_t$ . For example, if  $\alpha(p) = |p_1| + |p_2|$  for  $p = (p_1, p_2)$ , then the front  $\Gamma_1$  at  $t = 1$  is the closed curve shown in Fig. 1:

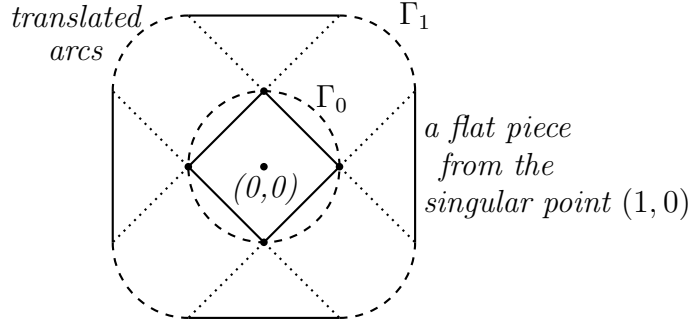


FIGURE 1. Front propagation and the shape of  $\Gamma_1$ .

### 3. THE PROOF OF THEOREM 1.1

Fix  $p \in \mathbb{R}^n$  to be an irrational vector satisfying a Diophantine condition, i.e., there exist  $c = c(p) > 0$  and  $\gamma > 0$  such that

$$|p \cdot k| \geq \frac{c}{|k|^\gamma} \quad \text{for all } k \in \mathbb{Z}^n \setminus \{0\}.$$

For small  $\varepsilon$ , let  $\overline{H}_\varepsilon(p)$  be the effective Hamiltonian associated with  $|p|^2 + \varepsilon V \cdot p$ , i.e.,

$$(3.11) \quad |p + Du^\varepsilon|^2 + \varepsilon V \cdot (p + Du^\varepsilon) = \overline{H}_\varepsilon(p).$$

We now perform a formal asymptotic expansion in term of  $\varepsilon$ , which will be proved rigorously by using the viscosity solution techniques. Suppose that

$$\begin{aligned} u^\varepsilon &= \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \cdots \\ \overline{H}_\varepsilon(p) &= a_0(p) + \varepsilon a_1(p) + \varepsilon^2 a_2(p) + \cdots \end{aligned}$$

We then get that

$$\begin{aligned}
 (3.12) \quad a_0(p) &= |p|^2 \\
 a_1(p) &= 2p \cdot D\phi_1 + V \cdot p \Rightarrow a_1(p) = 0 \\
 a_2(p) &= 2p \cdot D\phi_2 + |D\phi_1|^2 + V \cdot D\phi_1 \Rightarrow a_2(p) = \int_{\mathbb{T}^n} |D\phi_1|^2 dx \\
 &\dots
 \end{aligned}$$

Set

$$V = \sum_{k \neq 0} v_k e^{i2\pi k \cdot x}.$$

We need  $v_k \cdot k = 0$  for all  $k$  to have that  $\operatorname{div} V = 0$ . Then we get

$$D\phi_1 = -\frac{1}{2} \sum_{k \neq 0} \frac{(p \cdot v_k) e^{i2\pi k \cdot x} k}{p \cdot k},$$

and

$$a_2(p) = \frac{1}{4} \sum_{k \neq 0} \frac{|p \cdot v_k|^2 |k|^2}{|p \cdot k|^2}.$$

Thus, formally, we can conclude that

$$\overline{H}_\varepsilon(p) \approx |p|^2 + \varepsilon^2 \frac{1}{4} \sum_{k \neq 0} \frac{|p \cdot v_k|^2 |k|^2}{|p \cdot k|^2} + O(\varepsilon^3).$$

We now prove this expansion formula rigorously. See related computations in [15].

**Lemma 3.1.** *There exists  $\tau > 0$ , such that for all  $|p| \in [\tau, \frac{1}{\tau}]$ , we have*

$$(3.13) \quad \overline{H}_\varepsilon(p) = |p|^2 + \varepsilon^2 \frac{1}{4} \sum_{k \neq 0} \frac{|p \cdot v_k|^2 |k|^2}{|p \cdot k|^2} + O(\varepsilon^3)$$

as  $\varepsilon \rightarrow 0$ . Here, the error term satisfies  $|O(\varepsilon^3)| \leq K\varepsilon^3$  for some  $K$  depending only on  $\tau$ ,  $V$  and  $\frac{p}{|p|}$ .

*Proof.* As  $p$  satisfies a Diophantine condition, we are able to solve the following two equations explicitly in  $\mathbb{T}^n$  by computing Fourier coefficients

$$\begin{cases} p \cdot D\phi_1 = -\frac{1}{2} V \cdot p \\ p \cdot D\phi_2 = \frac{1}{2} (a_2(p) - |D\phi_1|^2 - V \cdot D\phi_1). \end{cases}$$

Here  $\phi_1, \phi_2 : \mathbb{T}^n \rightarrow \mathbb{R}$  are unknown functions.

Set  $w^\varepsilon = \varepsilon\phi_1 + \varepsilon^2\phi_2$ . Then, in light of the properties of  $\phi_1, \phi_2$ ,  $w^\varepsilon$  satisfies

$$|p + Dw^\varepsilon|^2 + \varepsilon V \cdot (p + Dw^\varepsilon) = |p|^2 + \varepsilon^2 a_2(p) + O(\varepsilon^3).$$

By looking at places where  $v^\varepsilon - w^\varepsilon$  attains its maximum and minimum and using the definition of viscosity solutions, we derive that

$$\overline{H}_\varepsilon(p) = |p|^2 + \varepsilon^2 a_2(p) + O(\varepsilon^3).$$

The error estimate can be read from the proof easily.  $\square$

It is obvious that  $a_2(p)$  is not a constant function of  $p$ . Hence Theorem 1.1 follows immediately from the following lemma.

**Lemma 3.2.** *For any vector  $p$  satisfying a Diophantine condition,*

$$(3.14) \quad \lim_{\varepsilon \rightarrow 0} \frac{\alpha_\varepsilon(p) - 2|p|}{\varepsilon^2|p|} = a_2(p).$$

*Proof.* Since  $\alpha_\varepsilon(p)$  is homogeneous of degree 1 and  $a_2(p)$  is homogeneous of degree 0 ( $a_2(p) = a_2(\lambda p)$  for all  $\lambda > 0$ ), we may assume that  $|p| = 1$ . Thanks to Lemma 3.1, we can write

$$(3.15) \quad \overline{H}_\varepsilon(p) = |p|^2 + \varepsilon^2 \frac{1}{4} \sum_{k \neq 0} \frac{|p \cdot v_k|^2 |k|^2}{|p \cdot k|^2} + O(\varepsilon^3) = 1 + \varepsilon^2 a_2(p) + O(\varepsilon^3).$$

Owing to Lemma 2.1, there exists a unique constant  $\lambda_\varepsilon = \lambda_\varepsilon(p) > 0$  such that

$$\alpha_\varepsilon(p) = \frac{1 + \overline{H}_\varepsilon(\lambda_\varepsilon p)}{\lambda_\varepsilon}.$$

It is easy to see that  $\lambda_\varepsilon \rightarrow 1$  as  $\varepsilon \rightarrow 0$ . So by Lemma 3.1 and  $\lambda_\varepsilon + \frac{1}{\lambda_\varepsilon} \geq 2$ ,

$$\alpha_\varepsilon(p) \geq 2 + \varepsilon^2 a_2(p) + O(\varepsilon^3).$$

Also, by the definition, it is obvious that

$$\alpha_\varepsilon \leq 1 + \overline{H}_\varepsilon(p) = 2 + \varepsilon^2 a_2(p) + O(\varepsilon^3).$$

Hence the conclusion of the lemma holds.  $\square$

#### 4. PROOFS OF THEOREMS 1.2 AND 1.3

Before proceeding to the proofs, we would like to point out something crucial in the arguments. Clearly, if the level curve  $\{\alpha = 1\}$  of  $\alpha$  is strictly convex at  $p$ , the level curve  $\{\overline{H} = \lambda_p - 1\}$  of  $\overline{H}$  must be strictly convex at  $\lambda_p p$ , where  $\lambda_p p$  is determined by Lemma 2.1. Nevertheless, our results do not follow from any rigidity result for  $\overline{H}$ , namely that of [6]. Indeed, different  $p$  in  $\{\alpha = 1\}$  might correspond to different  $\lambda_p$ , which corresponds to different energy levels of  $\overline{H}$ , but the rigidity result from [6] can be applied only on the same energy level. A key point of our proofs is to identify the exact location of at least one flat piece.

We first prove Claim (1) of Theorem 1.2.

**Proof of Theorem 1.2 (1).** We carry out the proof in a few steps.

**Step 1:** Due to Claim (2) of Lemma 2.1, the level curve  $S_\varepsilon$  is  $C^1$ . It is worth keeping in mind that  $\alpha_\varepsilon(p)$  is homogeneous of degree 1. For each  $p \in S_\varepsilon$ , denote  $n_p$  the outward unit normal vector at  $p$  to  $S_\varepsilon$ .

**Step 2:** If  $V$  is not constantly zero, by Lemma 4.1 below, there exists  $x_0 \in \mathbb{R}^2$  and a unit rational vector  $q_0$  and  $T > 0$  such that  $Tq_0 \in \mathbb{Z}^2$  and

$$\int_0^T q_0 \cdot DV(x_0 + q_0 t) dt \neq 0.$$

Here  $q \cdot DV = D(q \cdot V)$ .

**Step 3:** For each  $\varepsilon > 0$ , choose  $p_\varepsilon \in S_\varepsilon$  such that  $n_{p_\varepsilon} = q_0$ . To simplify notations, we write  $n_\varepsilon = n_{p_\varepsilon}$ . We claim that when  $\varepsilon$  is small enough,  $p_\varepsilon$  is a linear point of the set  $\{\alpha_\varepsilon = 1\}$ . Suppose this is false, then there exists a decreasing sequence  $\varepsilon_m \downarrow 0$

and as equence  $\{p_{\varepsilon_m}\}$  such that  $p_{\varepsilon_m}$  is not a linear point of the set  $\{\alpha_{\varepsilon_m} = 1\}$ . By (3) of Lemma 2.1,  $\tilde{p}_{\varepsilon_m} = \lambda_{\varepsilon_m} p_{\varepsilon_m}$  is not a linear point of the level curve  $\{\overline{H}_{\varepsilon_m} = \lambda_{\varepsilon_m} - 1\}$  either. Here  $\lambda_{\varepsilon_m} > 0$  is from Lemma 2.1. Clearly, the outward unit normal vector of the level curve  $\{\overline{H}_{\varepsilon_m} = \lambda_{\varepsilon_m} - 1\}$  at  $\tilde{p}_{\varepsilon_m}$  is also  $q_0$ . Accordingly to Theorem 2.3, the projected Mather set  $\mathcal{M}_{\tilde{p}_{\varepsilon_m}}$  is the whole torus  $\mathbb{T}^2$ . Moreover, by (2.6), there is a periodic minimizing orbit  $\xi_m : \mathbb{R} \rightarrow \mathbb{R}^2$  passing through  $x_0$  from Step 2 such that  $\xi_m(0) = x_0$ ,  $\xi_m(t_m) = x_0 + Tq_0$  for some  $t_m > 0$  and  $\xi$  satisfies the Euler-Lagrange equation associated with the Lagrangian  $L(q, x) = \frac{1}{4}|q - V|^2$ :

$$\frac{d(\dot{\xi}_m(t) - \varepsilon_m V(\xi_m(t)))}{dt} = -(\dot{\xi}_m(t) - \varepsilon_m V(\xi_m(t)) \cdot \varepsilon_m DV(\xi_m)).$$

Taking the integration on both sides over  $[0, t_m]$ , and by periodicity, we get

$$\int_0^{t_m} (\dot{\xi}_m(t) - \varepsilon_m V(\xi_m(t)) \cdot DV(\xi_m)) dt = 0.$$

Sending  $m \rightarrow +\infty$ , we find

$$\int_0^T q_0 \cdot DV(x_0 + q_0 t) dt = 0.$$

This contradicts to Step 2. As a result, we identified a flat piece of  $S_\varepsilon$ .  $\square$

**Lemma 4.1.** *If*

$$\int_0^T q \cdot DV(x + qt) dt = 0.$$

*for any  $x \in \mathbb{R}^2$  and any rational unit vector  $q \in \mathbb{R}^2$  and  $Tq \in \mathbb{Z}^2$ , then*

$$V \equiv 0$$

(Caution:  $q \cdot DV(x + qt) = D(q \cdot V(x + qt)) \neq \frac{dV(x+qt)}{dt}$ ).

*Proof.* Assume that  $V(y) = \sum_{k \in \mathbb{Z}^2} v_k e^{i2\pi k \cdot y}$ , where  $\{v_k\} \subset \mathbb{R}^2$  are Fourier coefficients of  $V$ . Since  $\text{div}(V) = 0$  and  $\int_{\mathbb{T}^2} V dx = 0$ , we have that  $v_0 = 0$  and

$$(4.16) \quad k \cdot v_k = 0$$

for all  $k$ . Then for any  $q \in \mathbb{R}^2$ ,

$$q \cdot DV(y) = D(q \cdot V) = 2\pi i \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} (q \cdot v_k) e^{i2\pi k \cdot y} k.$$

Now fix an integer vector  $k = (l_1, l_2) \in \mathbb{Z}^2 \setminus \{0\}$ . Choose  $q = (-l_2, l_1)$  and  $T = 1$ . Then

$$\int_0^1 q \cdot DV(x + qt) dt = 0 \quad \Rightarrow \quad q \cdot v_k e^{i2\pi k \cdot x} + q \cdot v_{-k} e^{-i2\pi k \cdot x} = 0$$

for any  $x \in \mathbb{R}^2$ . So  $q \cdot v_k = q \cdot v_{-k} = 0$ , which implies that  $v_k = \beta_k k$  for some  $\beta_k \in \mathbb{R}$ . Accordingly, (4.16) leads to  $\beta_k = 0$  for all  $k \in \mathbb{Z}^2$ . So  $V \equiv 0$ .  $\square$

Next we prove Claim (2) of Theorem 1.2.

**Proof of Theorem 1.2(2).** Recall that, for  $A \geq 0$  and  $p \in \mathbb{R}^2$ ,  $\alpha_A(p)$  is defined as

$$\alpha_A(p) = \inf_{\lambda > 0} \frac{\overline{H}_A(\lambda p) + 1}{\lambda}.$$

Here  $\overline{H}_A$  is the effective Hamiltonian associated with  $H_A(p, x) = |p|^2 + AV \cdot p$ . The corresponding Lagrangian is

$$L_A(q, x) = \frac{1}{4}|q - AV(x)|^2.$$

For  $p \in \mathbb{R}^2 \setminus \{0\}$  and  $A \geq 0$ , by Lemma 2.1, denote  $\lambda_{p,A} > 0$  as the unique positive number such that

$$\alpha_A(p) = \frac{\overline{H}_A(\lambda_{p,A}p) + 1}{\lambda_{p,A}}.$$

Let  $K$  be a stream function such that  $V = (-K_{x_2}, K_{x_1})$ . Clearly, we have that  $DK \cdot V \equiv 0$ . We consider the dynamical system  $\dot{\xi} = V(\xi)$ . By Poincaré recurrence theorem, we have the following two cases.

**Case 1:**  $\dot{\xi} = V(\xi)$  has a non-critical closed periodic orbit on  $\mathbb{R}^2$ . By the stability in 2d, there exists a strip of closed periodic orbits in its neighborhood. Without loss of generality, we may label them as  $\gamma_s(t)$  for  $s \in [0, \delta]$  for some  $\delta > 0$  sufficiently small such that  $K(\gamma_s(t)) \equiv s$  and  $\gamma_s(0) = \gamma_s(T_s)$  for some  $T_s > 0$  (minimum period). See the following figure. Denote  $\Gamma = \cup_{s \in [0, \delta]} \gamma_s$  and

$$\tau = \max_{x \in \Gamma} |DK(x)| > 0.$$

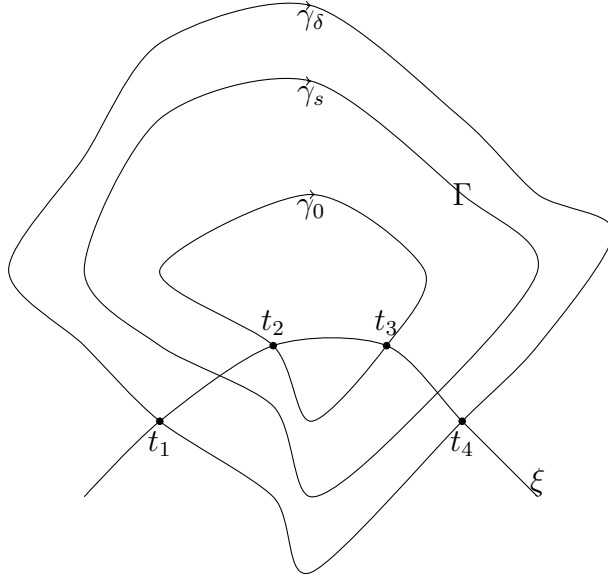


FIGURE 2. Closed periodic orbits in  $\Gamma$

**Claim 1:** For  $p \in \mathbb{R}^2$ , if  $\overline{H}_A(p) < \bar{c}A^2$  for  $\bar{c} = \frac{4\delta^2}{T_\delta^2 \tau^2}$ , then any unbounded absolutely minimizing trajectory associated with  $L_A + \overline{H}_A(p)$  cannot intersect  $\gamma_0$ .

We argue by contradiction. If not, let  $\xi : \mathbb{R} \rightarrow \mathbb{R}^2$  be an unbounded absolutely minimizing trajectory with  $\xi \cap \gamma_0 \neq \emptyset$ . Then there must exist  $t_1 < t_2 \leq t_3 < t_4$  such that

$$\xi(t_1), \xi(t_4) \in \gamma_\delta, \quad \xi(t_2), \xi(t_3) \in \gamma_0 \quad \text{and} \quad \xi([t_1, t_2]) \cup \xi([t_3, t_4]) \subset \Gamma.$$

See Figure 2 for demonstration. Set

$$E_1 = \int_{t_1}^{t_2} \frac{1}{4} |\dot{\xi} - AV(\xi)|^2 + \overline{H}_A(p) ds \quad \text{and} \quad E_2 = \int_{t_3}^{t_4} \frac{1}{4} |\dot{\xi} - AV(\xi)|^2 + \overline{H}_A(p) ds.$$

Since

$$|\dot{\xi} - AV(\xi)| \geq \frac{1}{\tau} |\dot{\xi} - AV(\xi)| \cdot |DK(\xi)| \geq \frac{1}{\tau} |(\dot{\xi} - AV(\xi)) \cdot DK(\xi)| = \frac{1}{\tau} |\dot{w}(t)|$$

for  $w(t) = K(\xi(t))$  and  $t \in [t_1, t_2] \cup [t_3, t_4]$ , we have that

$$\begin{aligned} E_1 + E_2 &\geq \overline{H}_A(p)(t_2 - t_1 + t_4 - t_3) + \frac{1}{4\tau^2} \left( \int_{t_1}^{t_2} |\dot{w}(t)|^2 dt + \int_{t_3}^{t_4} |\dot{w}(t)|^2 dt \right) \\ &\geq \overline{H}_A(p)(t_2 - t_1 + t_4 - t_3) + \frac{1}{4\tau^2} \left( \frac{\delta^2}{t_4 - t_3} + \frac{\delta^2}{t_2 - t_1} \right) \\ &\geq \frac{2\delta}{\tau} \sqrt{\overline{H}_A(p)}. \end{aligned}$$

However, if we travel from  $\xi(t_1)$  to  $\xi(t_4)$  along the route  $\gamma(s) = \gamma_\delta(sA)$ , the cost is at most  $\frac{T_\delta}{A} \overline{H}_A(p) < \frac{2\delta}{\tau} \sqrt{\overline{H}_A(p)}$ . This contradicts to the assumption that  $\xi$  is a minimizing trajectory. Hence our above claim holds.

Now choose  $\varepsilon_0 > 0$  such that  $\varepsilon_0^2 + \overline{M}\varepsilon_0 < \bar{c}$  for  $\overline{M} = \max_{\mathbb{T}^2} |V|$ . Owing to Lemma 4.3, there exists  $A_0$  such that if  $A \geq A_0$ , then

$$\lambda_{p,A} \leq \frac{\varepsilon_0}{2} A$$

for any unit vector  $p$ .

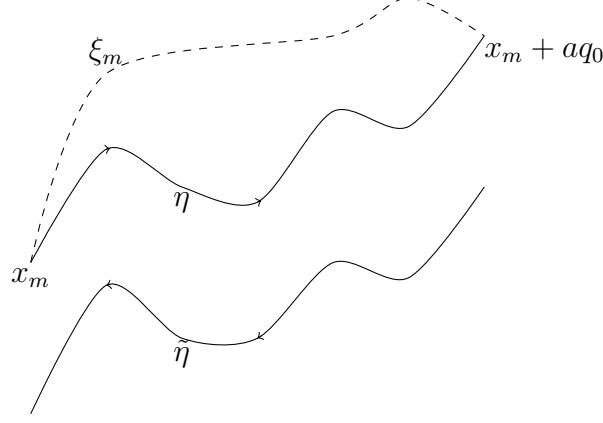
**Claim 2:** Assume that  $p_A \in S_A = \{\alpha_A = 1\}$  has a rational outward normal vector. Then  $p_A$  is a linear point of  $S_A$  if  $A \geq A_0$ .

It is equivalent to proving that  $\bar{p} = \frac{p_A}{|p_A|}$  is a linear point of the level curve  $\{\alpha_A(p) = \frac{1}{|p_A|}\}$ . We argue by contradiction. If not, by Theorem 2.3 and (3) of Lemma 2.1,  $\overline{H}_A$  is strictly convex at  $\lambda_{\bar{p}, A\bar{p}}$  and the associated projected Mather set  $\mathcal{M}_{\lambda_{\bar{p}, A\bar{p}}}$  is the whole Torus. Due to Claim 1, we must have that

$$\overline{H}_A(\lambda_{\bar{p}, A\bar{p}}) \geq \bar{c}A^2.$$

Since  $\overline{H}_A(p) \leq |p|^2 + A\overline{M}|p|$ , we have that  $\lambda_{A, \bar{p}} > \varepsilon_0 A$ . This contradicts to the choice of  $A$ . Therefore, the above claim holds and the result of this theorem follows.

**Case 2:** Next we consider the case when  $\dot{\xi} = V(\xi)$  has an unbounded periodic orbit  $\eta : \mathbb{R} \rightarrow \mathbb{R}^2$ , i.e.,  $\eta(T_0) - \eta(0) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$  for some  $T_0 > 0$ . Denote  $q_0$  as a rotation vector of  $\eta$ . Clearly,  $q_0$  is a rational vector. Since  $\int_{\mathbb{T}^2} V dx = 0$ , there also exists an unbounded periodic orbit  $\tilde{\eta}(t) = V(\tilde{\eta}(t))$  with a rotation vector  $-cq_0$  for some  $c > 0$ . See the following Figure 3.

FIGURE 3. Unbounded periodic orbits  $\eta$  and  $\tilde{\eta}$ 

**Claim 3:** Choose  $p_A \in S_A$  such that the unit outward normal vector at  $p_A$  is  $\frac{q_0}{|q_0|}$ . Then when  $A$  is large enough,  $p_A$  is a linear point of  $S_A$ . This is consistent with the last statement in Remark 1.1: when  $A$  is very large, we expect the shear structure to dominate the flame propagation.

It is equivalent to proving that  $\bar{p} = \frac{p_A}{|p_A|}$  is a linear point of the level curve  $\{\alpha_A(p) = \frac{1}{|p_A|}\}$  for sufficiently large  $A$ . We argue by contradiction. If not, then by (3) of Lemma 2.1, there exist a sequence  $A_m \rightarrow +\infty$  as  $m \rightarrow +\infty$  and  $|p_m| = 1$  such that  $\overline{H}_{A_m}(\lambda_m p_m)$  is strictly convex near  $\lambda_m p_m$ . Here  $\lambda_m > 0$  is the unique number satisfying (Lemma 2.1)

$$\alpha_{A_m}(p_m) = \frac{1 + \overline{H}_{A_m}(\lambda_m p_m)}{\lambda_m}.$$

Therefore, by Theorem 2.3, the associated projected Mather set  $\mathcal{M}_{\lambda_m p_m}$  is the whole torus. So there exists a unique periodic  $C^1$  solution  $v_m$  (up to additive constants) to

$$|\lambda_m p_m + Dv_m|^2 + A_m V \cdot (\lambda_m p_m + Dv_m) = \overline{H}_{A_m}(\lambda_m p_m) \quad \text{in } \mathbb{R}^2.$$

Let  $T_0$  and  $\tilde{T}_0$  be the minimal period of  $\eta$  and  $\tilde{\eta}$  respectively. Then  $q_0 = \frac{\eta(T_0) - \eta(0)}{T_0}$  and  $-cq_0 = \frac{\tilde{\eta}(T_0) - \tilde{\eta}(0)}{T_0}$ . Taking integration along  $\eta$  and  $\tilde{\eta}$ , we obtain that

$$\frac{1}{T_0} \int_0^{T_0} |\lambda_m p_m + Dv_m(\eta(s))|^2 ds + A_m q_0 \cdot \lambda_m p_m = \overline{H}_{A_m}(\lambda_m p_m)$$

and

$$\frac{1}{\tilde{T}_0} \int_0^{\tilde{T}_0} |\lambda_m p_m + Dv_m(\tilde{\eta}(s))|^2 ds - cA_m q_0 \cdot \lambda_m p_m = \overline{H}_{A_m}(\lambda_m p_m).$$

Accordingly, without loss of generality, we may assume that for all  $m \geq 1$ ,

$$\max_{s \in \mathbb{R}} |\lambda_m p_m + Dv_m(\eta(s))| \geq \sqrt{\overline{H}_{A_m}(\lambda_m p_m)}.$$

So there exists  $x_m \in \eta(\mathbb{R}) \cap [0, 1]^n$  such that

$$(4.17) \quad |\lambda_m p_m + Dv_m(x_m)| \geq \sqrt{\overline{H}_{A_m}(\lambda_m p_m)}.$$



Since the projected Mather set  $\mathcal{M}_{\lambda_m p_m}$  is the whole torus and the unit outward normal vector of  $\{\overline{H}_{A_m} = \overline{H}_{A_m}(\lambda_m p_m)\}$  at  $\lambda_m p_m$  is also  $\frac{q_0}{|q_0|}$ , by (2.6), we may find a periodic minimizing trajectory  $\xi_m : \mathbb{R} \rightarrow \mathbb{R}^2$  such that  $\xi_m(0) = x_m$  and  $\xi_m(t_m) = x_m + a_0 q_0$  (see Figure 3). Here  $t_m > 0$  is the minimal period of  $\xi_m$  and  $a_0 > 0$  is the smallest positive number such that  $a_0 q_0 \in \mathbb{Z}^2$ . Note that  $\eta(T_0) - \eta(0) = a_0 q_0$  as well. Moreover, by (2.4),

$$(4.18) \quad \dot{\xi}_m = 2(\lambda_m p_m + Dv_m(\xi_m)) + A_m V(\xi_m)$$

and

$$\frac{d(\dot{\xi}_m(s) - A_m V(\xi_m(s)))}{ds} = -(\dot{\xi}_m - A_m V(\xi_m)) A_m DV(\xi_m).$$

Since  $\xi_m$  is an absolutely minimizing trajectory, we must have that

$$(4.19) \quad \frac{T_0}{A_m} \overline{H}_{A_m}(\lambda_m p_m) \geq \int_0^{t_m} \frac{1}{4} |\dot{\xi}_m(s) - A_m V(\xi_m)|^2 ds + t_m \overline{H}_{A_m}(\lambda_m p_m).$$

The left hand side of the above is the cost of traveling along the route  $\gamma(s) = \eta(s A_m)$  from  $x_m$  to  $x_m + a q_0$ . So  $t_m \leq \frac{T_0}{A_m}$ . Consider

$$w_m(s) = \xi_m\left(\frac{s}{A_m}\right).$$

Then  $w_m$  is a periodic curve with a minimal period  $A_m t_m \leq T_0$ ,

$$(4.20) \quad \frac{1}{4} |\dot{w}_m - V(w_m)|^2 + \frac{1}{2} V(w_m) \cdot (\dot{w}_m - V(w_m)) = \frac{\overline{H}_{A_m}(\lambda_m p_m)}{A_m^2}$$

and

$$\frac{d(\dot{w}_m(s) - V(w_m(s)))}{ds} = -(\dot{w}_m(s) - V(w_m(s))) \cdot DV(w_m(s)).$$

Hence there exists a constant  $\theta_0 > 0$  depending only on  $V$  such that

$$(4.21) \quad \frac{\min_{s \in \mathbb{R}} |\dot{w}_m(s) - V(w_m(s))|}{\max_{s \in \mathbb{R}} |\dot{w}_m(s) - V(w_m(s))|} > \theta_0.$$

Owing to (4.19), we obtain that

$$(4.22) \quad \int_0^{A_m t_m} |\dot{w}_m(s) - V(w_m(s))|^2 ds \leq \frac{4 T_0 \overline{H}_{A_m}(\lambda_m p_m)}{A_m^2}.$$

Owing to (4.20) and Lemma 4.3,  $\max_{s \in \mathbb{R}} |\dot{w}_m(s)|$  is uniformly bounded. Since  $w_m(A_m t_m) - w_m(0) = a_0 q_0$ , it is clear that

$$\liminf_{m \rightarrow +\infty} A_m t_m > 0.$$

Now combining (4.22), (4.21) and Lemma 4.3, it is not hard to show that

$$\lim_{m \rightarrow +\infty} A_m t_m = T_0$$

and

$$(4.23) \quad \lim_{m \rightarrow +\infty} w_m(s) = \eta(s) \quad \text{uniformly in } C^1(\mathbb{R}^1).$$

Write  $c_m = \max_{s \in \mathbb{R}} |\dot{w}_m(s) - V(w_m(s))|$ . Note that

$$\dot{w}_m(s) = \frac{2(\lambda_m p_m + Dv_m(w_m(s)))}{A_m} + V(w_m(s)).$$

and by (4.20),

$$V(w_m) \cdot \frac{\dot{w}_m - V(w_m)}{c_m} = \frac{2\overline{H}_{A_m}(\lambda_m p_m)}{A_m^2 c_m} - \frac{1}{2c_m} |\dot{w}_m - V(w_m)|^2.$$

Due to (4.17) and (4.18),  $c_m A_m \geq 2\sqrt{\overline{H}_{A_m}(\lambda_m p_m)}$ . Combining with (4.23) and Lemma 4.3, we have that

$$(4.24) \quad \lim_{m \rightarrow +\infty} V(w_m(s)) \cdot \frac{\dot{w}_m(s) - V(w_m(s))}{c_m} = 0 \quad \text{uniformly in } \mathbb{R}^1.$$

Note that

$$\frac{dK(w_m(s))}{ds} = DK(w_m(s)) \cdot (\dot{w}_m(s) - V(w_m(s))).$$

Taking integration from 0 to  $t_m A_m$ , due to periodicity, we have that

$$\int_0^{t_m A_m} DK(w_m(s)) \cdot \left( \frac{\dot{w}_m(s) - V(w_m(s))}{c_m} \right) ds = 0.$$

By (4.21),  $\frac{|\dot{w}_m(s) - V(w_m(s))|}{c_m} \in [\theta_0, 1]$ . Combining with (4.24), by sending  $m \rightarrow +\infty$ , we obtain that

$$\int_0^{T_0} a(s) |DK(\eta(s))| ds = 0$$

for some  $a(t) > 0$ . This is a contradiction. So our claim holds.

Combining Case 1 and Case 2, we obtain the desired result.  $\square$

**Lemma 4.2.** *Let  $\overline{H}$  be the effective Hamiltonian of*

$$|p + Dv|^2 + V(x) \cdot (p + Dv) = \overline{H}(p).$$

*Then for  $|p| \geq \theta > 0$ , there exists  $\mu_\theta > 0$  depending only on  $\theta$  and  $V$  such that*

$$\min_{q \in \partial \overline{H}(p)} q \cdot p \geq \overline{H}(p) + \mu_\theta.$$

*Proof.* This follows easily from a compactness argument,  $\overline{H}(0) = 0$ ,  $\overline{H}(p) \geq |p|^2$  and the strict convexity of  $\overline{H}$  along the radial direction (Theorem 2.2).  $\square$

Due to the simple equality  $\overline{H}(p) = \frac{\overline{H}_A(Ap)}{A^2}$ , we immediately derive the following corollary.

**Corollary 4.1.** *If  $|p| \geq \theta A$ , then*

$$\min_{q \in \partial \overline{H}_A(p)} q \cdot p \geq \overline{H}_A(p) + \mu_\theta A^2.$$

**Lemma 4.3.** For  $|p| = 1$  and  $A \geq 1$ , denote  $\lambda_{p,A}$  such that

$$\alpha_A(p) = \frac{\overline{H}_A(\lambda_{p,A}p) + 1}{\lambda_{p,A}}.$$

Then

$$\lim_{A \rightarrow +\infty} \frac{\max_{|p|=1} \lambda_{p,A}}{A} = \lim_{A \rightarrow +\infty} \frac{\max_{|p|=1} \overline{H}_A(\lambda_{p,A}p)}{A^2} = 0.$$

*Proof.* Since  $\overline{H}_A(p) \leq |p|^2 + A\overline{M}|p|$  for  $\overline{M} = \max_{\mathbb{T}^2} |V|$ , the second limit holds true immediately once we prove the validity of the first limit.

We prove the first limit by contradiction. If not, then there exists a sequence  $A_m \rightarrow +\infty$  as  $m \rightarrow +\infty$  and  $|p_m| = 1$  such that for  $\lambda_m = \lambda_{p_m, A_m}$ ,

$$\lim_{m \rightarrow +\infty} \frac{\lambda_m}{A_m} = b_0 > 0.$$

So by Lemma 2.1, there is  $q_m \in \partial \overline{H}_{A_m}(\lambda_m p_m)$  such that

$$q_m \cdot \lambda_m p_m = \overline{H}_{A_m}(\lambda_m p_m) + 1.$$

This contradicts to Corollary 4.1 when  $m$  is large enough.  $\square$

Finally, we prove Theorem 1.3.

**Proof of Theorem 1.3.** It suffices to show that there exists a unit vector  $p_0$  such that  $sp_0$  is a linear point of  $\{\overline{H}_A = \overline{H}_A(sp_0)\}$  for any  $s > 0$ .

(1) Assume that  $V$  is the shear flow, i.e.  $V = (v(x_2), 0)$ . Without loss of generality, we omit the dependence on  $A$ . It is easy to see that  $\overline{H}_A(p)$  has the following explicit formulas: for  $p = (p_1, p_2) \in \mathbb{R}^2$ ,

$$\overline{H}(p) = |p_1|^2 + h(p_1, p_2)$$

and  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$  is given by

$$\begin{cases} h(p) = M(p_1) = \max_{y \in \mathbb{T}} p_1 v(y) & \text{if } |p_2| \leq \int_0^1 \sqrt{M(p_1) - p_1 v(y)} dy \\ |p_2| = \int_0^1 \sqrt{h(p) - p_1 v(y)} dy & \text{otherwise.} \end{cases}$$

Hence  $\overline{H}$  is linear near the point  $p = (s, 0)$  as long as  $s \neq 0$ .

(2) Now let  $V = (-K_{x_2}, K_{x_1})$  for  $K(x) = \sin 2\pi x_1 \sin 2\pi x_2$ . Recall that the cell problem is

$$(4.25) \quad |p + Dv|^2 + AV \cdot (p + Dv) = \overline{H}_A(p) \geq |p|^2.$$

The proof for the cellular flow case of Theorem 1.3 follows directly from the result of the following proposition.  $\square$

**Proposition 4.1.** Fix  $s > 0$  and let  $Q = (s, 0) \in \mathbb{R}^2$ . If  $A \neq 0$ , then  $Q$  is a linear point of the level set  $\{\overline{H}_A = \overline{H}_A(Q)\}$ .

*Proof.* Let  $\mathcal{M}_Q$  be the projected Mather set at the point  $Q$ . By symmetry, it is easy to see that  $\partial\overline{H}(Q)$  is parallel to  $(1,0)$ . Then due to Theorem 2.3, it suffices to show that

$$(4.26) \quad \mathcal{M}_Q \cap \{y \in \mathbb{T}^2 : y_2 = 0\} = \emptyset.$$

**Step 1:** We claim that there is a viscosity solution  $v$  to (4.25) which satisfies that  $v(y_1, y_2) = v(y_1, -y_2)$ . In fact, let us now look at the discounted approximation of (4.25) with  $p = Q$ . For each  $\varepsilon > 0$ , consider

$$(4.27) \quad \varepsilon v^\varepsilon + |Q + Dv^\varepsilon|^2 + AV \cdot (Q + Dv^\varepsilon) = 0 \quad \text{in } \mathbb{T}^2,$$

which has a unique viscosity solution  $v^\varepsilon \in C^{0,1}(\mathbb{T}^2)$ . By the fact that  $Q = (s, 0)$  and the special structure of  $V$ , it is clear that  $(y_1, y_2) \mapsto v^\varepsilon(y_1, -y_2)$  is also a solution to the above. Therefore,  $v^\varepsilon(y_1, y_2) = v^\varepsilon(y_1, -y_2)$  for all  $(y_1, y_2) \in \mathbb{T}^2$ . Clearly, any convergent subsequence of  $v^\varepsilon - v^\varepsilon(0)$  tends to a  $v$  which is a solution of (4.25) and is even in the  $y_2$  variable. We would like to point out that a recent result of Davini, Fathi, Iturriaga and Zavidovique [8] (see also Mitake and Tran [21]) gives the convergence of the full sequence  $v^\varepsilon - v^\varepsilon(0)$  as  $\varepsilon \rightarrow 0$ .

**Step 2:** Assume by contradiction that (4.26) is not correct. Suppose that

$$(\mu_0, 0) \in \mathcal{M}_Q \cap \{y \in \mathbb{T}^2 : y_2 = 0\}.$$

Then  $v$  is differentiable at  $(\mu_0, 0)$  and  $v_{x_2}(\mu_0, 0) = 0$ . Due to (2.4), the flow-invariance of the Mather set and the Euler-Lagrangian equation, it is easy to see that

$$\{y \in \mathbb{T}^2 : y_2 = 0\} \subset \mathcal{M}_Q.$$

Hence  $v$  is  $C^1$  along the  $y_1$  axis and

$$(4.28) \quad v_{y_2}(y_1, 0) = 0 \quad \text{for all } y_1 \in \mathbb{T}.$$

Set  $w(y_1) = v(y_1, 0)$ . Plug this into the equation (4.25) of  $v$  with  $y_2 = 0$  and use (4.28) to get that

$$(4.29) \quad |s + w'|^2 - A(s + w') \sin(2\pi y_1) = \overline{H}_A(Q) \geq s^2 \quad \text{in } \mathbb{T}.$$

Clearly,  $s + w'(y_1) \neq 0$  for all  $y_1 \in \mathbb{T}$  in light of (4.29). Note further that  $w' \in C(\mathbb{T})$  and

$$\int_0^1 (s + w'(y_1)) dy_1 = s > 0.$$

Thus,  $s + w' > 0$  in  $\mathbb{T}$  and for all  $y_1 \in \mathbb{T}$ ,

$$s + w'(y_1) = \frac{1}{2} \left( A \sin(2\pi y_1) + \sqrt{A^2 \sin^2(2\pi y_1) + 4\overline{H}_A(Q)} \right).$$

Integrate this over  $\mathbb{T}$  to deduce that

$$\begin{aligned} s &= \int_0^1 (s + w'(y_1)) dy_1 = \int_0^1 \frac{1}{2} \left( A \sin(2\pi y_1) + \sqrt{A^2 \sin^2(2\pi y_1) + 4\overline{H}_A(Q)} \right) dy_1 \\ &= \int_0^1 \frac{1}{2} \sqrt{A^2 \sin^2(2\pi y_1) + 4\overline{H}_A(Q)} dy_1 \geq \sqrt{\overline{H}_A(Q)} \geq s. \end{aligned}$$

Therefore, all inequalities in the above must be equalities. In particular, the second last inequality must be an equality, which yields that  $A = 0$ .  $\square$

**Remark 4.1.** *Theorem 2.3 is not really necessary to get the above proposition. In fact, using the same argument, we can derive that the Aubry set has no intersection with the  $y_1$  axis. Then by [14], there is a strict subsolution to (4.25) near the  $y_1$  axis. The linearity of  $\overline{H}$  near  $Q$  will follow from some elementary calculations.*

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